

Sample covariance matrices of heavy-tailed distributions

Konstantin Tikhomirov

Department of Math. and Stat. Sciences, University of Alberta, Canada

Email: ktikhomi@ualberta.ca

June 14, 2016

Abstract

Let $p > 2$, $B \geq 1$, $N \geq n$ and let X be a centered n -dimensional random vector with the identity covariance matrix such that $\sup_{a \in S^{n-1}} \mathbb{E}|\langle X, a \rangle|^p \leq B$. Further, let X_1, X_2, \dots, X_N be independent copies of X , and $\Sigma_N := \frac{1}{N} \sum_{i=1}^N X_i X_i^T$ be the sample covariance matrix. We prove that

$$K^{-1} \|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2} \leq \frac{1}{N} \max_{i \leq N} \|X_i\|^2 + \left(\frac{n}{N}\right)^{1-2/p} \log^4 \frac{N}{n} + \left(\frac{n}{N}\right)^{1-2/\min(p,4)}$$

with probability at least $1 - \frac{1}{n}$, where $K > 0$ depends only on B and p . In particular, for all $p > 4$ we obtain a quantitative Bai–Yin type theorem.

1 Introduction

Estimation of the covariance matrix of a multidimensional distribution is a standard problem in statistics. Assume we have a centered n -dimensional random vector X with an unknown covariance matrix $\Sigma = \mathbb{E}XX^T$, and N independent copies of X (a sample): X_1, X_2, \dots, X_N . In general, the problem is to construct an estimator for Σ — a function of X_1, X_2, \dots, X_N taking values in the set of $n \times n$ matrices, such that for certain class of distributions the random matrix produced by the estimator is close (in some sense) to the actual covariance matrix Σ . Various restrictions may be imposed on the distribution of X . Recent developments in the subject showed that, under certain assumptions on the moments of 1-dimensional projections of X , together with some rather strong structural assumptions on Σ , it is possible to obtain a satisfactory estimator of Σ even when the size of the sample N is much smaller than the dimension n . There is a vast literature dealing with these questions, which, however, do not have direct connection with our results. As an example of those developments, we refer to [4, 5].

In this note, we consider the standard estimator — *the sample covariance matrix*, defined as $\Sigma_N := \frac{1}{N} \sum_{i=1}^N X_i X_i^T = \frac{1}{N} A_N^T A_N$, where A_N is the $N \times n$ random matrix with rows X_1, X_2, \dots, X_N . The law of large numbers implies that for any distribution

with a well-defined covariance matrix we have the convergence $\Sigma_N \xrightarrow[N \rightarrow \infty]{\text{a.s.}} \Sigma$ entry-wise, hence, in any operator norm. The question is what size of the sample one should take to approximate the actual covariance matrix by Σ_N with a given precision and probability.

Given a random vector X , by F_X we shall denote the cdf of X . Let \mathcal{F} be a class of n -dimensional centered distributions which is closed under invertible linear transformations (i.e. $F_{T(X)} \in \mathcal{F}$ whenever $F_X \in \mathcal{F}$ and $T \in \text{GL}_n(\mathbb{R})$), and let $\delta \in (0, 1)$. We want to identify the number N such that for any $F_X \in \mathcal{F}$ with the covariance matrix Σ , and for corresponding sample X_1, X_2, \dots, X_N we have $\|\Sigma_N - \Sigma\|_{2 \rightarrow 2} \leq \delta \|\Sigma\|_{2 \rightarrow 2}$ with probability close to one, where $\|\cdot\|_{2 \rightarrow 2}$ denotes the spectral norm of a matrix, i.e. its largest singular value. It can be easily shown that the question reduces to checking the relation $\|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2} \leq \delta$ for *isotropic* distributions from \mathcal{F} i.e. those having the identity covariance matrix. Moreover, the last inequality is equivalent to

$$\sqrt{(1 - \delta)N} \leq s_{\min}(A_N) \leq s_{\max}(A_N) \leq \sqrt{(1 + \delta)N},$$

where $s_{\min}(A_N)$ and $s_{\max}(A_N)$ are the smallest and the largest singular values of A_N given by $s_{\min}(A_N) := \inf_{a \in S^{n-1}} \|A_N(a)\|$, $s_{\max}(A_N) := \sup_{a \in S^{n-1}} \|A_N(a)\|$.

Both limiting and non-limiting properties of the extreme singular values of A_N have received considerable attention from researchers. Let us refer to the classical works [33] and [3] regarding almost sure convergence of appropriately normalized singular values when the coordinates of the underlying distributions are i.i.d.; as well as more recent works [22, 14, 18, 25, 9, 1, 2, 29, 19, 26, 20, 17, 28, 11, 10, 31, 32, 21, 27, 8]. For a more comprehensive list of results, we refer to surveys [24] and [30]. Let us remark that the question of approximating the covariance matrix for *log-concave* distributions appeared in geometric functional analysis in connection with the problem of computing the volume of a convex set given by a separation oracle (see [15]). That question was considered, in particular, in [7, 22] and was completely resolved in [1, 2].

The purpose of this note is to establish approximation properties of the sample covariance matrix under very mild assumptions on the distribution. Fix for a moment any $p > 2$ and $B \geq 1$. Assume that X is a centered n -dimensional random vector with a covariance matrix Σ . Assume that X satisfies:

$$\mathbb{E}|\langle X, a \rangle|^p \leq B(\mathbb{E}\langle X, a \rangle^2)^{p/2} = B\langle a, \Sigma a \rangle^{p/2} \text{ for all } a \in \mathbb{R}^n.$$

The set of all distributions F_X for centered random vectors X satisfying the above condition will be denoted by $\mathcal{F}(n, p, B)$. It is not difficult to check that the class $\mathcal{F}(n, p, B)$ is closed under invertible linear transformations (in the sense discussed above). For isotropic distributions, the above condition is simplified to $\mathbb{E}|\langle X, a \rangle|^p \leq B$ for all $a \in S^{n-1}$.

The main result of the note is the following theorem:

Theorem 1. *There is a non-increasing function $\nu : (2, \infty) \rightarrow \mathbb{R}_+$ with the following property: Let $p > 2$, $B \geq 1$, and assume that $N \geq 2n$. Further, let X be a centered n -dimensional random vector with covariance matrix Σ , whose distribution belongs to the class $\mathcal{F}(n, p, B)$. Let X_1, X_2, \dots, X_N be independent copies of X , and let $\Sigma_N = \frac{1}{N} \sum_{i=1}^N X_i X_i^T$. Then the sample covariance matrix Σ_N satisfies*

$$\nu(p)^{-1} \frac{\|\Sigma_N - \Sigma\|_{2 \rightarrow 2}}{\|\Sigma\|_{2 \rightarrow 2}} \leq \frac{1}{N} \max_{i \leq N} \langle X_i, \Sigma^{-1} X_i \rangle + B^{2/p} \left(\frac{n}{N} \right)^{\frac{p-2}{p}} \log^4 \frac{N}{n} + B^{2/p} \left(\frac{n}{N} \right)^{\frac{\min(p, 4) - 2}{\min(p, 4)}}$$

with probability at least $1 - \frac{1}{n}$.

Note that the right-hand side of the above expression depends on the precision matrix Σ^{-1} . As an additional assumption on the distribution, one can make sure that $\frac{1}{N} \max_{i \leq N} \langle X_i, \Sigma^{-1} X_i \rangle$ is typically smaller by the order of magnitude than the remaining summands. Such an assumption implies that X is concentrated in the norm $\sqrt{\langle \cdot, \Sigma^{-1} \cdot \rangle}$. As an example, assuming that $\|\Sigma^{-1}\|_{2 \rightarrow 2}, \|\Sigma\|_{2 \rightarrow 2} \leq C'$ and $\|X\| \leq C\sqrt{n}$ with very large probability for some constants $C, C' > 0$, we get $\langle X, \Sigma^{-1} X \rangle \leq C' \|X\|^2 \leq C' C^2 n$ with high probability, so that the summand $\frac{1}{N} \max_{i \leq N} \langle X_i, \Sigma^{-1} X_i \rangle$ can be disregarded.

For $p \in (2, 4]$, the last summand in the estimate of Theorem 1 is dominated by the second one, and we can rewrite the conclusion of the theorem as

$$\mathbb{P}\left\{\nu(p)^{-1} \frac{\|\Sigma_N - \Sigma\|_{2 \rightarrow 2}}{\|\Sigma\|_{2 \rightarrow 2}} \leq \frac{1}{N} \max_{i \leq N} \langle X_i, \Sigma^{-1} X_i \rangle + B^{2/p} \left(\frac{n}{N}\right)^{\frac{p-2}{p}} \log^4 \frac{N}{n}\right\} \geq 1 - \frac{1}{n}.$$

On the other hand, since $\log^4 \frac{N}{n}$ grows with N slower than any positive power of $\frac{N}{n}$, for $p > 4$ we can essentially disregard the second summand in the estimate of Theorem 1. Let us provide a separate statement, which we formulate for isotropic distributions.

Corollary 2. *There is a non-increasing function $\tilde{\nu} : (4, \infty) \rightarrow \mathbb{R}_+$ with the following property: Let $p > 4$, $B \geq 1$, and assume that $N \geq 2n$. Let X be a centered isotropic vector with $\sup_{a \in \mathbb{S}^{n-1}} \mathbb{E}|\langle X, a \rangle|^p \leq B$, and let X_1, X_2, \dots, X_N be its independent copies. Then the sample covariance matrix $\Sigma_N = \frac{1}{N} \sum_{i=1}^N X_i X_i^T$ satisfies*

$$\mathbb{P}\left\{\tilde{\nu}(p)^{-1} \|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2} \leq \frac{1}{N} \max_{i \leq N} \|X_i\|^2 + B^{2/p} \sqrt{\frac{n}{N}}\right\} \geq 1 - \frac{1}{n}.$$

In case when the coordinates of the random vector X are i.i.d. centered random variables with a bounded fourth moment, the well known result of Z.D. Bai and Y.Q. Yin [3] implies that

$$\|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2} = O\left(\sqrt{\frac{n}{N}}\right)$$

with probability close to one. In this connection, Corollary 2 can be viewed as a Bai–Yin type estimate for quite general class of distributions.

Let us make some further remarks. For $p > 2$, $B \geq 1$ and for all isotropic distributions from $\mathcal{F}(n, p, B)$, Theorem 1 provides the following bound for the extreme singular values of matrix A_N :

$$\begin{aligned} N - K \max_{i \leq N} \|X_i\|^2 - KN \left(\frac{n}{N}\right)^{\frac{p-2}{p}} \log^4 \frac{N}{n} &\leq s_{\min}(A_N)^2 \\ &\leq s_{\max}(A_N)^2 \leq 1 + K \max_{i \leq N} \|X_i\|^2 + KN \left(\frac{n}{N}\right)^{\frac{p-2}{p}} \log^4 \frac{N}{n}, \quad \text{if } p \leq 4, \end{aligned}$$

and

$$\begin{aligned} N - K \max_{i \leq N} \|X_i\|^2 - K\sqrt{nN} &\leq s_{\min}(A_N)^2 \\ &\leq s_{\max}(A_N)^2 \leq N + K \max_{i \leq N} \|X_i\|^2 + K\sqrt{nN}, \quad \text{if } p > 4 \end{aligned}$$

with probability $\geq 1 - \frac{1}{n}$, where $K = K(p, B) > 0$ depends only on p, B . Note that better estimates for the *smallest* singular value were previously obtained in [17] and later strengthened in [31, 32]. The papers [26] and [17] were apparently the first ones where lower bounds for the smallest singular value were given in quite a general setting without any restrictions on the magnitude of the matrix norm $\|A_N\|_{2 \rightarrow 2}$. The novelty of our work consists in proving the upper bound for the largest singular value. This problem has been extensively studied in the literature. In [1, 2], an analog of Theorem 1 was proved for distributions with sub-exponential tails of one-dimensional projections. In paper [26], just “ $2 + \varepsilon$ ” moment assumptions were employed, but, as an additional requirement, the authors assumed certain tail decay for *all* projections (of any rank) of the random vector. In [20], an equivalent of Theorem 1 was proved under $8 + \varepsilon$ moment assumption, and, finally, in [10], the result of [20] was extended to $p > 4$, however the authors of [10] did not obtain a Bai–Yin type estimate in the regime $4 < p \leq 8$.

Thus, our input is two-fold: *first, we extend the theorems of [20] and [10] to the range $p > 2$, and, second, in the regime $p > 4$ we obtain a Bai–Yin type estimate for the largest singular value.* The factor “ $\log^4 \frac{N}{n}$ ” in the second summand of our bound, which comes into play in the regime $2 < p \leq 4$, seems excessive. We believe that some essential new arguments are required to completely eliminate the log-factor, if it is at all possible.

As another illustration, let us consider a particular form of the above theorem, which provides an estimate for the spectral norm of a square random matrix with i.i.d. columns under very mild assumptions on the distribution:

Theorem 3. *Let $p > 2$ and $B \geq 1$. Then there exist $K_1 = K_1(p, B)$ and $K_2 = K_2(p, B)$ depending only on p and B with the following property: Let $n \geq K_1$ and let A be an $n \times n$ random matrix with i.i.d. columns Y_1, Y_2, \dots, Y_n , where each Y_i is a centered isotropic random vector satisfying $\sup_{a \in S^{n-1}} \mathbb{E} |\langle Y_i, a \rangle|^p \leq B$. Then the spectral norm of A can be estimated as*

$$\|A\|_{2 \rightarrow 2} \leq K_2 \max_{i \leq N} \|Y_i\|$$

with probability at least $1 - \frac{4}{n}$.

The core of the proof of Theorem 1 is a “chaining” argument for quadratic forms already employed in [20, 10]. At the same time, two crucial new ingredients are added: First, we define a “coloring” of the sample, which is essentially a truncation procedure for the inner products of the sample vectors. Second is a Sparsifying Lemma, which allows to significantly decrease cardinalities of ε -nets constructed in the proof, thereby providing better probabilistic estimates for quadratic forms. The Sparsifying Lemma allowed us to get the Bai–Yin type estimate for $p > 4$, and together with the coloring technique, to extend the range of admissible p ’s to $(2, \infty)$.

The structure of the paper is the following: In Section 2, we collect the notation and several auxiliary lemmas. In Section 3, we define the coloring of the sample. In Sections 4 and 5, we define and estimate certain quadratic forms. In particular, the Sparsifying Lemma (Lemma 11) is given in Section 4. Finally, in Section 6, we complete the proof of the main result.

2 Preliminaries

The set of natural numbers will be denoted by \mathbb{N} , and reals — by \mathbb{R} . Given a natural number k , $[k]$ is the set $\{1, 2, \dots, k\}$. Cardinality of a finite set S will be denoted by $|S|$. For a real number a , $\lfloor a \rfloor$ is the largest integer not exceeding a , whereas $\lceil a \rceil$ is the smallest integer greater or equal to a . Let S^{N-1} be the standard unit sphere in \mathbb{R}^N and $\{e_i\}_{i=1}^N$ be the standard basis vectors in \mathbb{R}^N . For brevity, for any subset $I \subset [N]$, by \mathbb{R}^I we denote the span of the vectors $\{e_i\}_{i \in I}$. Given a vector $y \in \mathbb{R}^N$, by $|y| \in \mathbb{R}_+^N$ we denote the vector of the absolute values of coordinates of y .

The standard inner product in \mathbb{R}^N will be denoted by $\langle \cdot, \cdot \rangle$, and the canonical Euclidean norm — by $\|\cdot\|$. For a vector $v \in \mathbb{R}^N$, $\|v\|_p$ ($1 \leq p \leq \infty$) is the standard ℓ_p^N norm. For a matrix M , its spectral norm is denoted by $\|M\|_{2 \rightarrow 2}$.

Given a real non-negative sequence $(a_i)_{i=1}^N$, a subset $J \subset [N]$ and $k \in \mathbb{N}$, denote by $(k)\text{-}\max_{\ell \in J} a_\ell$ the k -th largest element of the subsequence $(a_i)_{i \in J}$. When $k > |J|$, we set $(k)\text{-}\max_{\ell \in J} a_\ell := 0$.

Given a graph $G = (V, E)$, a vertex coloring of G is an assignment of “colors” to all vertices such that no adjacent vertices share the same color. The smallest possible number of colors sufficient to assign a vertex coloring for G is called *the chromatic number of G* and is denoted by $\chi(G)$.

For any $\rho > 0$ and a subset $S \subset \mathbb{R}^N$, a *Euclidean ρ -net* \mathcal{N} in S is any subset of S such that for every $x \in S$ there is $y \in \mathcal{N}$ with $\|x - y\| \leq \rho$. If, additionally, one can always find $y \in \mathcal{N}$ with $\text{supp } y \subset \text{supp } x$ and $\|x - y\| \leq \rho$ then we will call \mathcal{N} a *support-preserving ρ -net*.

A vector $y \in \mathbb{R}^N$ is *r -sparse* (for some $r \geq 0$) if $|\text{supp } y| \leq r$. The following lemma can be proved by standard arguments:

Lemma 4. *For every $\rho \in (0, 1]$ and any natural $r \leq N$, there exists a support-preserving ρ -net \mathcal{N} in the set of all r -sparse unit vectors in \mathbb{R}^N of cardinality at most $(\frac{C_4 N}{\rho r})^r$. Here, $C_4 > 0$ is a universal constant.*

The next lemma, stated in [10] (the argument appeared already in [2]), will be very helpful for us.

Lemma 5 ([10, Lemma 4.1]). *Let M be an $n \times n$ matrix, $\rho \in (0, 1/2)$, and let \mathcal{N} be a Euclidean ρ -net in S^{n-1} . Then*

$$\sup_{y \in S^{n-1}} |\langle My, y \rangle| \leq (1 - 2\rho)^{-1} \sup_{z \in \mathcal{N}} |\langle Mz, z \rangle|.$$

Next, we recall two well known inequalities regarding the distribution of sums of independent random variables.

Lemma 6 (W.Hoeffding, [13]). *Let $\xi_1, \xi_2, \dots, \xi_m$ be independent random variables, such that $\xi_i \in [a_i, b_i]$ a.s. for some numbers $a_i, b_i \in \mathbb{R}$ ($i = 1, 2, \dots, m$). Then*

$$\mathbb{P}\left\{\sum_{i=1}^m \xi_i - \sum_{i=1}^m \mathbb{E}\xi_i \geq mt\right\} \leq \exp\left(-2m^2 t^2 / \sum_{i=1}^m (b_i - a_i)^2\right), \quad t > 0.$$

Given a random variable ξ , its *Lévy concentration function* $\mathcal{Q}(\xi, \cdot)$ is defined as

$$\mathcal{Q}(\xi, t) = \sup_{\lambda \in \mathbb{R}} \mathbb{P}\{|\xi - \lambda| \leq t\}, \quad t \geq 0.$$

Lemma 7 (H.Kesten, [16]). *Let $\xi_1, \xi_2, \dots, \xi_m$ be independent random variables, and let $0 < a_1, a_2, \dots, a_m \leq 2R$ be some real numbers. Then*

$$\mathcal{Q}\left(\sum_{j=1}^m \xi_j, R\right) \leq C_7 R \frac{\sum_{j=1}^m a_j^2 (1 - \mathcal{Q}(\xi_j, a_j)) \mathcal{Q}(\xi_j, R)}{\left(\sum_{j=1}^m a_j^2 (1 - \mathcal{Q}(\xi_j, a_j))\right)^{3/2}}.$$

Here, $C_7 > 0$ is a universal constant.

The next lemma provides an elementary estimate of order statistics for a set of independent non-negative variables.

Lemma 8. *Let $h \geq 1$, $B \geq 1$, $r \in \mathbb{N}$ and let $\xi_1, \xi_2, \dots, \xi_r$ be independent non-negative random variables such that $\mathbb{E}\xi_i^h \leq B$, $i = 1, 2, \dots, r$. Then for any $m \leq r$ and $\tau > 0$ we have*

$$\mathbb{P}\left\{(m)\text{-}\max_{\ell \in [r]} \xi_\ell \geq \tau\right\} \leq \left(\frac{eBr}{\tau^h m}\right)^m.$$

Proof. We have

$$\mathbb{P}\left\{(m)\text{-}\max_{\ell \in [r]} \xi_\ell \geq \tau\right\} \leq \binom{r}{m} \left(\frac{B}{\tau^h}\right)^m \leq \left(\frac{eBr}{\tau^h m}\right)^m.$$

□

A centered random vector X in \mathbb{R}^n is *isotropic* if its covariance matrix $\mathbb{E}XX^T$ is the identity. Let us give a simple bound for the norm of an isotropic vector assuming certain moment conditions on its one-dimensional projections:

Lemma 9. *Let X be a centered n -dimensional isotropic vector, and suppose that for some $p > 2$ and $B \geq 1$ we have*

$$\sup_{y \in S^{n-1}} \mathbb{E}|\langle X, y \rangle|^p \leq B.$$

Then for any $\tau > 0$ we have

$$\mathbb{P}\{\|X\| \geq \tau\} \leq Bn^{p/2}\tau^{-p}.$$

Proof. Note that $\|X\| \leq n^{1/2-1/p}\|X\|_p$ (deterministically), whence

$$\mathbb{E}\|X\|^p \leq n^{p/2-1}\mathbb{E}\|X\|_p^p \leq Bn^{p/2}.$$

Then, by Markov's inequality,

$$\mathbb{P}\{\|X\| \geq \tau\} \leq Bn^{p/2}\tau^{-p}.$$

□

3 Coloring the sample

Let X_1, X_2, \dots, X_N be the i.i.d. copies of a centered n -dimensional isotropic vector X . Further, fix a number $H > 0$. We construct a random undirected graph \mathcal{G}_H with the vertex set $[N]$ by defining its edge set as

$$\{(i, j) : 1 \leq i < j \leq N, |\langle X_i, X_j \rangle| > H \max_{h \leq N} \|X_h\|\}.$$

Let $\chi(\mathcal{G}_H)$ be the chromatic number of the graph. In what follows, for each $H > 0$ we define a random partition $\{\mathcal{C}_m^H\}_{m=1}^N$ of $[N]$, measurable with respect to the σ -algebra generated by X_1, X_2, \dots, X_N , and satisfying the following two conditions:

- 1) $\mathcal{C}_m^H = \emptyset$ for all $m > \chi(\mathcal{G}_H)$;
- 2) For any $m \leq N$ and $i, j \in \mathcal{C}_m^H$ with $i \neq j$, the vertices i and j are not adjacent within \mathcal{G}_H , i.e. $|\langle X_i, X_j \rangle| \leq H \max_{h \leq N} \|X_h\|$.

The collection $\{\mathcal{C}_m^H\}_{m=1}^N$ will be called *the coloring of the sample X_1, X_2, \dots, X_N with threshold H* . Such a coloring will act as a way to “truncate” the inner products $\langle X_i, X_j \rangle$ and will be employed when estimating quadratic forms in Sections 4 and 5. At a more technical level, our estimate of the largest eigenvalue of the matrix $\sum_{i=1}^N X_i X_i^T$ involves expressions $\log(n) \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle|$ for some subsets $\mathcal{C} \subset [N]$ (see Proposition 14, where they appear first time). A trivial upper bound $\log(n) \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \leq \log(n) \max_{i \leq N} \|X_i\|^2$ is not useful here; instead we build the argument in a way that produces an upper bound of the form

$$\lambda_{\max} \left(\sum_{i=1}^N X_i X_i^T \right) \lesssim \log(n) \sum_{m=1}^N \max_{i \neq j \in \mathcal{C}_m^H} |\langle X_i, X_j \rangle| + \dots \leq \chi(\mathcal{G}_H) H \max_{i \leq N} \|X_i\| + \dots$$

Then a proper definition of H , together with a control of the random quantity $\chi(\mathcal{G}_H)$, give a satisfactory estimate for λ_{\max} . In a sense, we partition the original sample X_1, X_2, \dots, X_N into several subsets in such a way that within each subset the vectors are “almost” pairwise orthogonal.

The next statement provides tail bounds for the chromatic number $\chi(\mathcal{G}_H)$:

Proposition 10. *Assume that for some $p > 2$ and $B \geq 1$ we have $\mathbb{E}|\langle X, y \rangle|^p \leq B$ for all $y \in S^{n-1}$. Then for any $H > 0$ and any integer $m > 1$ the chromatic number of \mathcal{G}_H satisfies $\chi(\mathcal{G}_H) \leq m$ with probability at least $1 - (BNH^{-p})^m n^{p/2}$.*

Proof. Let us introduce an auxiliary random process $Y(i)$ on $[N]$ with values in \mathbb{N} , where $Y(1) := 1$ (constant) and for all $i = 2, 3, \dots, N$:

$$Y(i) := \min\{r \in \mathbb{N} : \forall j < i (j \in \mathbb{N}) \text{ with } Y(j) = r \text{ we have } |\langle X_i, X_j \rangle| \leq H \|X_j\|\}.$$

Note that by the very definition of $Y(i)$, we have that any two numbers $i \neq j \in [N]$ such that $Y(i) = Y(j)$, are not adjacent in \mathcal{G}_H ; in particular, $\chi(\mathcal{G}_H) \leq \max_{i \in [N]} Y(i)$. Next, for

each $i > 1$ and $m \geq 1$ we have

$$\begin{aligned}
& \mathbb{P}\{Y(i) = m + 1\} \\
& \leq \mathbb{P}\{\exists \ell \leq i - 1 \text{ such that } |\langle X_i, X_\ell \rangle| > H\|X_\ell\| \text{ and } Y(\ell) = m\} \\
& \leq \sum_{\ell=1}^{i-1} \mathbb{P}\{|\langle X_i, X_\ell \rangle| > H\|X_\ell\| \text{ and } Y(\ell) = m\} \\
& \leq BH^{-p} \sum_{\ell=1}^{i-1} \mathbb{P}\{Y(\ell) = m\} \\
& \leq BH^{-p} \mathbb{E}|\{j \leq N : Y(j) = m\}|.
\end{aligned}$$

Hence,

$$\mathbb{E}|\{j \leq N : Y(j) = m + 1\}| \leq BNH^{-p} \mathbb{E}|\{j \leq N : Y(j) = m\}|.$$

Next, in view of Lemma 9,

$$\begin{aligned}
\mathbb{E}|\{j \leq N : Y(j) = 2\}| & \leq \mathbb{E}|\{j \leq N : \|X_j\| > H\}| \\
& = N\mathbb{P}\{\|X_1\| > H\} \\
& \leq BNH^{-p}n^{p/2}.
\end{aligned}$$

Combining the estimates, we obtain for every $m \geq 1$:

$$\mathbb{E}|\{j \leq N : Y(j) = m + 1\}| \leq (BNH^{-p})^m n^{p/2}.$$

Note that the set of values $\{Y(j) : j \leq N\}$ is an interval in \mathbb{N} , whence

$$\begin{aligned}
\mathbb{P}\{\chi(\mathcal{G}_H) \geq m + 1\} & \leq \mathbb{P}\{\exists j \leq N \text{ with } Y(j) = m + 1\} \\
& \leq \mathbb{E}|\{j \leq N : Y(j) = m + 1\}| \\
& \leq (BNH^{-p})^m n^{p/2}.
\end{aligned}$$

□

4 Quadratic forms — Deterministic estimates

As before, let X be a centered random vector in \mathbb{R}^n and X_1, X_2, \dots, X_N be its independent copies. Additionally, we assume that the covariance matrix of X is the identity. By A_N we denote the $N \times n$ random matrix with rows X_1, X_2, \dots, X_N . For every natural $k \leq N$ and any subset $\mathcal{C} \subset [N]$, denote

$$f(k, \mathcal{C}) := \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ \text{supp } y \subset \mathcal{C}, \\ |\text{supp } y| \leq k}} \left\| \sum_{i=1}^N y_i X_i \right\|^2. \quad (1)$$

Obviously, $f(N, [N]) = \|A_N\|_{2 \rightarrow 2}^2$. Later, we will take \mathcal{C} to be one of classes from the coloring defined in the previous section, in particular, \mathcal{C} will be a *random* set depending

on X_1, X_2, \dots, X_N . In this section, we will not estimate probabilities of any events, but instead produce deterministic estimates for $f(k, \mathcal{C})$ as well as other quantities considered below. The next relations provide a basis for our analysis. We have

$$\begin{aligned} f(k, \mathcal{C}) &\leq \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ \text{supp } y \subset \mathcal{C}, \\ |\text{supp } y| \leq k}} \sum_{i=1}^N y_i^2 \|X_i\|^2 + \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ \text{supp } y \subset \mathcal{C}, \\ |\text{supp } y| \leq k}} \sum_{i \neq j} y_i y_j \langle X_i, X_j \rangle \\ &\leq \max_{i \leq N} \|X_i\|^2 + \sup_{\substack{y, z \in \mathbb{S}^{N-1}, \\ \text{supp } y, \text{supp } z \subset \mathcal{C}, \\ |\text{supp } y|, |\text{supp } z| \leq k}} \sum_{i \neq j} y_i z_j \langle X_i, X_j \rangle. \end{aligned}$$

Next, denoting $I^c := [N] \setminus I$ for any $I \subset [N]$, we get:

$$\begin{aligned} \sup_{\substack{y, z \in \mathbb{S}^{N-1}, \\ \text{supp } y, \text{supp } z \subset \mathcal{C}, \\ |\text{supp } y|, |\text{supp } z| \leq k}} \sum_{i \neq j} y_i z_j \langle X_i, X_j \rangle &= 2^{-N+2} \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ \text{supp } y \subset \mathcal{C}, \\ |\text{supp } y| \leq k}} \sup_{\substack{z \in \mathbb{S}^{N-1}, \\ \text{supp } z \subset \mathcal{C}, \\ |\text{supp } z| \leq k}} \sum_{I \subset [N]} \langle \sum_{i \in I} y_i X_i, \sum_{j \in I^c} z_j X_j \rangle \\ &\leq 2^{-N+2} \sum_{I \subset [N]} \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k}} \sup_{\substack{z \in \mathbb{S}^{N-1}, \\ |\text{supp } z| \leq k}} \langle \sum_{i \in I \cap \mathcal{C}} y_i X_i, \sum_{j \in I^c \cap \mathcal{C}} z_j X_j \rangle. \end{aligned}$$

For each $I \subset [N]$, denote

$$g(k, \mathcal{C}, I) := \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k}} \sup_{\substack{z \in \mathbb{S}^{N-1}, \\ |\text{supp } z| \leq k}} \langle \sum_{i \in I \cap \mathcal{C}} y_i X_i, \sum_{j \in I^c \cap \mathcal{C}} z_j X_j \rangle. \quad (2)$$

Further, for every vector $v \in \mathbb{R}^N$ and any $i \leq N$ we set

$$W_{v,i} := \langle X_i, \sum_{j=1}^N v_j X_j \rangle. \quad (3)$$

We recall that for any sequence $(a_\ell)_{\ell \in I^c \cap \mathcal{C}}$ of non-negative real numbers, by (j) - $\max_{\ell \in I^c \cap \mathcal{C}} a_\ell$ we denote the j -th largest element of the sequence. Then we have for any integer $m \leq k$:

$$\begin{aligned} g(k, \mathcal{C}, I) &= \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \sup_{\substack{z \in \mathbb{S}^{N-1}, \\ |\text{supp } z| \leq k, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} \sum_{j=1}^N z_j W_{y,j} \\ &= \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \left(\sum_{j=1}^k (j)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} W_{y,\ell}^2 \right)^{1/2} \\ &\leq \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \left(\sum_{j=1}^m (j)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} W_{y,\ell}^2 \right)^{1/2} + \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \left(\sum_{j=m+1}^k (j)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} W_{y,\ell}^2 \right)^{1/2} \\ &\leq \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \sup_{\substack{z \in \mathbb{S}^{N-1}, \\ |\text{supp } z| \leq m, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} \langle \sum_{i=1}^N y_i X_i, \sum_{j=1}^N z_j X_j \rangle + \sqrt{k} \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} (m)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}|. \end{aligned}$$

Further,

$$\begin{aligned}
& \sup_{\substack{y \in S^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq m, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} \left\langle \sum_{i=1}^N y_i X_i, \sum_{j=1}^N z_j X_j \right\rangle \\
&= \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq m, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} \sup_{\substack{y \in S^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \sum_{j=1}^N y_j W_{z,j} \\
&= \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq m, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} \left(\sum_{j=1}^k (j) \cdot \max_{\ell \in I \cap \mathcal{C}} W_{z,\ell}^2 \right)^{1/2} \\
&\leq \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq m, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} \sup_{\substack{y \in S^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} \left\langle \sum_{i=1}^N y_i X_i, \sum_{j=1}^N z_j X_j \right\rangle + \sqrt{k} \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq m, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} (m) \cdot \max_{\ell \in I \cap \mathcal{C}} |W_{z,\ell}|.
\end{aligned}$$

Thus, we can write for $m \leq k$:

$$\begin{aligned}
g(k, \mathcal{C}, I) &\leq g(m, \mathcal{C}, I) + \\
&\quad \sqrt{k} \sup_{\substack{y \in S^{N-1}, \\ |\text{supp } y| \leq k, \\ \text{supp } y \subset I \cap \mathcal{C}}} (m) \cdot \max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}| + \sqrt{k} \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq k, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} (m) \cdot \max_{\ell \in I \cap \mathcal{C}} |W_{z,\ell}|. \tag{4}
\end{aligned}$$

Let us remark that estimates for quadratic forms similar to the ones above, appeared in literature before. In particular, we refer to a work of J. Bourgain [7], which deals with approximating covariance matrices of log-concave distributions (see also [1]), as well as papers [19, 20] where a “chaining” argument was employed for dealing with heavy-tailed distributions (see [10] for further development of the technique).

Unlike the above computations, the next lemma is a new addition to the arguments employed in [20, 10]. It provides a “sparsifying” technique which will allow us to considerably decrease cardinalities of ε -nets involved in the proof and, as a result, weaken the moment assumptions on the distributions.

Lemma 11 (Sparsifying Lemma). *Let $\delta \in (0, 1]$, $k \geq 12/\delta^2$, $m \geq 4$, and let $T = (t_{ij})$ be an $m \times k$ matrix of reals. Then for any $y \in S^{k-1}$ there is a coordinate projection $P : \mathbb{R}^k \rightarrow \mathbb{R}^k$ of rank $\text{rank } P \leq \delta k$ such that*

$$C_{11}^{-1} \delta^2 \min_{\ell \leq m} |Ty|_\ell \leq \frac{\max_{i,j} |t_{ij}|}{\sqrt{k}} + (\lfloor m/4 \rfloor) \cdot \max_{\ell \in [m]} |TP(y)|_\ell.$$

Here, $C_{11} > 0$ is a universal constant, and $(\lfloor m/4 \rfloor) \cdot \max_{\ell \in [m]} |TP(y)|_\ell$ is the $\lfloor m/4 \rfloor$ -th largest coordinate of the vector $|TP(y)| \in \mathbb{R}^m$.

Arguments similar in spirit to Lemma 11, and based on Maurey’s empirical method, have been recently employed to verify RIP properties of the Fourier matrices (see, in

particular, [23, 6, 12]). Let us note that the dependence on δ of the left-hand side of the bound in Lemma 11 can probably be improved, decreasing the power of the logarithmic factor in the estimate from Theorem 1; however, will not eliminate it completely.

Proof of Lemma 11. Fix a vector $y \in \mathbb{S}^{k-1}$. Without loss of generality, we can assume that all coordinates of y are non-negative, and that $\min_{i \leq m} \left| \sum_{j=1}^k t_{ij} y_j \right| > 0$. Denote

$$J := \{j \leq k : y_j \geq 2/\sqrt{\delta k}\}.$$

It is easy to see that $|J| \leq \delta k/4$. Consider two cases:

1) The set

$$I := \left\{ i \leq m : \left| \sum_{j=1}^k t_{ij} y_j \right| > 2 \left| \sum_{j \in J} t_{ij} y_j \right| \right\}$$

has cardinality less than $m/2$. Then, taking P to be the orthogonal projection onto the span of $\{e_j\}_{j \in J}$, we get

$$\left| \sum_{j=1}^k t_{ij} y_j \right| \leq 2 \left| \sum_{j \in J} t_{ij} y_j \right| = 2 |TP(y)|_i$$

for all $i \in I^c := [m] \setminus I$ (with $|I^c| \geq m/2$), implying the statement.

2) The set I has cardinality at least $m/2$. For brevity, let us denote $J^c := [k] \setminus J$. First, assume that for some $i_0 \in I$ and $j_0 \in J^c$ we have $|t_{i_0 j_0} y_{j_0}| \geq \frac{\delta \sqrt{\delta}}{8eC_7} \left| \sum_{j \in J^c} t_{i_0 j} y_j \right|$. Then, in view of the definition of J , we get

$$\max_{i,j} |t_{ij}| \geq |t_{i_0 j_0}| \geq \frac{\delta^2 \sqrt{k}}{16eC_7} \left| \sum_{j \in J^c} t_{i_0 j} y_j \right| \geq \frac{\delta^2 \sqrt{k}}{32eC_7} \left| \sum_{j=1}^k t_{i_0 j} y_j \right| \geq \frac{\delta^2 \sqrt{k}}{32eC_7} \min_{i \leq m} \left| \sum_{j=1}^k t_{ij} y_j \right|,$$

implying the statement. For the rest of the proof, we will suppose that

$$|t_{iq} y_q| < \frac{\delta \sqrt{\delta}}{8eC_7} \left| \sum_{j \in J^c} t_{ij} y_j \right| \quad \text{for all } q \in J^c \text{ and } i \in I. \quad (5)$$

Define a random coordinate projection $\tilde{P} : \mathbb{R}^k \rightarrow \mathbb{R}^k$ as follows: Let $\{\eta_j\}_{j \in J^c}$ be i.i.d. Bernoulli (0-1) random variables with probability of success $\delta/2$, and set $\text{Im} \tilde{P} := \text{span}\{\eta_j e_j\}_{j \in J^c}$. Clearly, $\text{rank} \tilde{P} = \sum_{j \in J^c} \eta_j$, and by Hoeffding's inequality (Lemma 6) we have $\mathbb{P}\{\text{rank} \tilde{P} > \delta k\} \leq \exp(-\delta^2 k/4) \leq 0.1$. We will show that for any index $i \in I$ we have

$$|T\tilde{P}(y)|_i = \left| \sum_{j \in J^c} \eta_j t_{ij} y_j \right| \geq \frac{\delta}{4} \left| \sum_{j \in J^c} t_{ij} y_j \right| \geq \frac{\delta}{8} \left| \sum_{j=1}^k t_{ij} y_j \right|$$

with probability at least $1 - \exp(-1)$. Fix any $i \in I$. First, assume that

$$\left(\sum_{j \in J^c} t_{ij} y_j\right)^2 \geq \frac{8}{\delta^2} \sum_{j \in J^c} t_{ij}^2 y_j^2.$$

Then by Hoeffding's inequality (Lemma 6), we have

$$\begin{aligned} \mathbb{P}\left\{\left|\sum_{j \in J^c} \eta_j t_{ij} y_j\right| < \frac{\delta}{4} \left|\sum_{j \in J^c} t_{ij} y_j\right|\right\} &\leq \exp\left(-\frac{\delta^2}{8} \left(\sum_{j \in J^c} t_{ij} y_j\right)^2 / \sum_{j \in J^c} t_{ij}^2 y_j^2\right) \\ &\leq \exp(-1). \end{aligned}$$

Now, assume that $\left(\sum_{j \in J^c} t_{ij} y_j\right)^2 < \frac{8}{\delta^2} \sum_{j \in J^c} t_{ij}^2 y_j^2$. Then, applying Kesten's inequality (Lemma 7) with $R := \frac{\delta\sqrt{\delta}}{8eC_7} \left|\sum_{j \in J^c} t_{ij} y_j\right|$ and $a_j := \frac{1}{2}|t_{ij} y_j|$ (note that $2R \geq a_j$ for all $j \in J^c$ in view of (5)), we obtain

$$\begin{aligned} \mathcal{Q}\left(\sum_{j \in J^c} \eta_j t_{ij} y_j, R\right) &\leq C_7 R \left(\sum_{j \in J^c} a_j^2 (1 - \mathcal{Q}(\eta_j t_{ij} y_j, a_j))\right)^{-1/2} \\ &\leq \sqrt{\frac{8}{\delta}} C_7 R \left(\sum_{j \in J^c} t_{ij}^2 y_j^2\right)^{-1/2} \\ &\leq \frac{8C_7 R}{\delta\sqrt{\delta}} \left|\sum_{j \in J^c} t_{ij} y_j\right|^{-1} \\ &\leq \exp(-1). \end{aligned}$$

Thus, for any $i \in I$ we have

$$\mathbb{P}\left\{\left|\sum_{j \in J^c} \eta_j t_{ij} y_j\right| < \frac{\delta\sqrt{\delta}}{8eC_7} \left|\sum_{j \in J^c} t_{ij} y_j\right|\right\} \leq \exp(-1),$$

whence, by the definition of I ,

$$\mathbb{P}\left\{\left|\sum_{j \in J^c} \eta_j t_{ij} y_j\right| \geq \frac{\delta\sqrt{\delta}}{16eC_7} \left|\sum_{j=1}^k t_{ij} y_j\right| \text{ and } \text{rank} \tilde{P} \leq \delta k\right\} > \frac{1}{2}, \quad i \in I$$

(recall that $\mathbb{P}\{\text{rank} \tilde{P} \leq \delta k\} \geq 0.9$). This immediately implies that there is a (non-random) realization P of \tilde{P} such that $\text{rank} P \leq \delta k$, and

$$|TP(y)|_i \geq \frac{\delta\sqrt{\delta}}{16eC_7} \left|\sum_{j=1}^k t_{ij} y_j\right|$$

for at least half of the indices $i \in I$, i.e. for at least $m/4$ indices. The result follows. \square

The next statement is an application of Lemma 11 to relation (4):

Lemma 12. *Let $\delta \in (0, 1]$, $k \geq 12/\delta^2$ and $4 \leq m \leq k$. Further, let $I, \mathcal{C} \subset [N]$ be subsets of $[N]$ (whether fixed or random). Then*

$$g(k, \mathcal{C}, I) \leq g(m, \mathcal{C}, I) + 2C_{11}\delta^{-2} \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \\ + C_{11}\sqrt{k}\delta^{-2} \sup_{\substack{y \in S^{N-1}, \\ |\text{supp} y| \leq \delta k, \\ \text{supp} y \subset I \cap \mathcal{C}}} (\lfloor m/4 \rfloor)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}| + C_{11}\sqrt{k}\delta^{-2} \sup_{\substack{z \in S^{N-1}, \\ |\text{supp} z| \leq \delta k, \\ \text{supp} z \subset I^c \cap \mathcal{C}}} (\lfloor m/4 \rfloor)\text{-}\max_{\ell \in I \cap \mathcal{C}} |W_{z,\ell}|,$$

where g and W are defined in (2) and (3).

Proof. Fix a realization of the vectors X_1, X_2, \dots, X_N and of the sets I, \mathcal{C} , and consider the quantity

$$\sup_{\substack{y \in S^{N-1}, \\ |\text{supp} y| \leq k, \\ \text{supp} y \subset I \cap \mathcal{C}}} (m)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}|.$$

Without loss of generality, we can assume that it is non-zero. Let $\tilde{y} \in S^{N-1}$ with $|\text{supp} \tilde{y}| \leq k$ and $\text{supp} \tilde{y} \subset I \cap \mathcal{C}$ be a vector which delivers the supremum in the above expression. Note that necessarily $|I^c \cap \mathcal{C}| \geq m$. Let $U \subset I^c \cap \mathcal{C}$ be a set of indices ℓ of cardinality m corresponding to m largest elements of the sequence $(|W_{\tilde{y},\ell}|)_{\ell \in I^c \cap \mathcal{C}}$, and let $V := \text{supp} \tilde{y} \subset I \cap \mathcal{C}$. Then we define an $m \times |V|$ matrix $T = (t_{\ell j})$ whose elements are the inner products $\langle X_\ell, X_j \rangle$ ($\ell \in U, j \in V$). For convenience, we index the elements of the matrix over the Cartesian product $U \times V$. Then we can define the multiplication $T\tilde{y}$ in a natural way by setting $T\tilde{y} := (W_{\tilde{y},\ell})_{\ell \in U}$. Note that

$$\min_{\ell \in U} \left| \sum_{j \in V} t_{\ell j} \tilde{y}_j \right| = (m)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{\tilde{y},\ell}|.$$

Then, applying Lemma 11, we get that there is a coordinate projection $P : \mathbb{R}^V \rightarrow \mathbb{R}^V$ of rank at most δk such that

$$C_{11}^{-1}\delta^2 (m)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{\tilde{y},\ell}| \leq \frac{\max_{\ell,j} |t_{\ell j}|}{\sqrt{k}} + (\lfloor m/4 \rfloor)\text{-}\max_{\ell \in U} |TP(\tilde{y})|_\ell \\ \leq \frac{\max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle|}{\sqrt{k}} + (\lfloor m/4 \rfloor)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{P\tilde{y},\ell}|.$$

Hence,

$$\sup_{\substack{y \in S^{N-1}, \\ |\text{supp} y| \leq k, \\ \text{supp} y \subset I \cap \mathcal{C}}} (m)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}| \leq \frac{C_{11} \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle|}{\delta^2 \sqrt{k}} + C_{11}\delta^{-2} \sup_{\substack{y \in S^{N-1}, \\ |\text{supp} y| \leq \delta k, \\ \text{supp} y \subset I \cap \mathcal{C}}} (\lfloor m/4 \rfloor)\text{-}\max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}|.$$

Repeating the argument for

$$\sup_{\substack{z \in S^{N-1}, \\ |\text{supp} z| \leq k, \\ \text{supp} z \subset I^c \cap \mathcal{C}}} (m)\text{-}\max_{\ell \in I \cap \mathcal{C}} |W_{z,\ell}|,$$

and applying relation (4), we obtain the statement. \square

The next lemma is a variation of the standard procedure of passing from supremum over a set of vectors to the supremum over a net.

Lemma 13. *Let $\rho \in (0, 1]$, $r, h, p, q \in \mathbb{N}$ with $r \geq 2$, and let \mathcal{N} be a support-preserving Euclidean ρ -net on the set of all h -sparse unit vectors in \mathbb{R}^q . Further, let $T = (t_{ij})$ be any $p \times q$ matrix. Then*

$$\sup_{\substack{u \in S^{q-1}, \\ |\text{supp} u| \leq h}} (r)\text{-}\max_{\ell \in [p]} |Tu|_\ell \leq 2 \sup_{v \in \mathcal{N}} (\lfloor r/2 \rfloor)\text{-}\max_{\ell \in [p]} |Tv|_\ell + \frac{4\rho}{\sqrt{r}} \sup_{\substack{u \in S^{q-1}, \\ |\text{supp} u| \leq h}} \left(\sum_{i=1}^r (i)\text{-}\max_{\ell \in [p]} |Tu|_\ell^2 \right)^{1/2}.$$

Proof. Without loss of generality, $r \leq p$. Fix a vector $\tilde{u} \in S^{q-1}$ with $|\text{supp} \tilde{u}| \leq h$. By the definition of \mathcal{N} , there is $\tilde{v} \in \mathcal{N}$ with $\text{supp} \tilde{v} \subset \text{supp} \tilde{u}$ and $\|\tilde{u} - \tilde{v}\| \leq \rho$. Assume that $2(\lfloor r/2 \rfloor)\text{-}\max_{\ell \in [p]} |T\tilde{v}|_\ell < (r)\text{-}\max_{\ell \in [p]} |T\tilde{u}|_\ell$. Let σ be a permutation on p elements such that the sequence $(|T\tilde{u}|_{\sigma(i)}, 1 \leq i \leq p)$ is non-increasing. Then the last condition implies that there is a subset $J \subset [r]$ of cardinality at least $r/2$ such that $|T\tilde{u}|_{\sigma(i)} > 2|T\tilde{v}|_{\sigma(i)}$ for all $i \in J$, implying that $|T(\tilde{u} - \tilde{v})|_i \geq \frac{1}{2}(r)\text{-}\max_{\ell \in [p]} |T\tilde{u}|_\ell$ for at least $r/2$ indices $i \in [p]$. At the same time, $\|\tilde{u} - \tilde{v}\| \leq \rho$ and $|\text{supp}(\tilde{u} - \tilde{v})| \leq h$. Setting $s := \frac{\tilde{u} - \tilde{v}}{\|\tilde{u} - \tilde{v}\|}$, it follows that

$$\sup_{\substack{u \in S^{q-1}, \\ |\text{supp} u| \leq h}} \left(\sum_{i=1}^r (i)\text{-}\max_{\ell \in [p]} |Tu|_\ell^2 \right)^{1/2} \geq \left(\sum_{i=1}^r (i)\text{-}\max_{\ell \in [p]} |Ts|_\ell^2 \right)^{1/2} \geq \frac{1}{2\rho} \sqrt{\frac{r}{2}} (r)\text{-}\max_{\ell \in [p]} |T\tilde{u}|_\ell.$$

Thus, we have shown that for any vector $\tilde{u} \in S^{q-1}$ with $|\text{supp} \tilde{u}| \leq h$ we have

$$(r)\text{-}\max_{\ell \in [p]} |T\tilde{u}|_\ell \leq 2 \sup_{v \in \mathcal{N}} (\lfloor r/2 \rfloor)\text{-}\max_{\ell \in [p]} |Tv|_\ell + \frac{4\rho}{\sqrt{r}} \sup_{\substack{u \in S^{q-1}, \\ |\text{supp} u| \leq h}} \left(\sum_{i=1}^r (i)\text{-}\max_{\ell \in [p]} |Tu|_\ell^2 \right)^{1/2}.$$

Taking the supremum over all admissible \tilde{u} , we get the result. \square

Combining Lemmas 12 and 13, we obtain the main result of the section:

Proposition 14. *Let $I \subset [N]$ be a fixed subset; $\mathcal{C} \subset [N]$ be random, and let $\delta \in (0, 1/3)$, with $k \geq 24/\delta^2$ and $N \geq 128C_{14}\delta^{-2}k$. Denote $t := \lfloor \log_2 \frac{\delta^2 k}{24} \rfloor$ and define $k_j := \lfloor k/2^j \rfloor$, $0 \leq j \leq t$. Then there are subsets of δk_j -sparse unit vectors \mathcal{N}_j and \mathcal{N}'_j ($0 \leq j \leq t-1$) supported on I and I^c , respectively, such that 1) $|\mathcal{N}_j|, |\mathcal{N}'_j| \leq \left(\frac{C_4 N}{\delta k_j}\right)^{2\delta k_j}$ for all admissible j , and 2) we have*

$$\begin{aligned} g(k, \mathcal{C}, I) &\leq C_{14}\delta^{-2} \log(k) \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \\ &\quad + C_{14}\delta^{-2} \sum_{j=0}^{t-1} \sqrt{k_j} \sup_{u \in \mathcal{N}_j} (\lfloor k_{j+1}/16 \rfloor)\text{-}\max_{\ell \in I^c} |W_{u,\ell}| \\ &\quad + C_{14}\delta^{-2} \sum_{j=0}^{t-1} \sqrt{k_j} \sup_{v \in \mathcal{N}'_j} (\lfloor k_{j+1}/16 \rfloor)\text{-}\max_{\ell \in I} |W_{v,\ell}|. \end{aligned}$$

Here, $C_{14} > 0$ is a universal constant, and g and W are defined by (2) and (3).

Proof. First, we fix any $j < t$ and consider the quantity $g(k_j, \mathcal{C}, I)$. We define $\mathcal{N}_j \subset \mathbb{R}^I$ as a support-preserving $\frac{k_j}{N}$ -net in the set of δk_j -sparse unit vectors in \mathbb{R}^I , of cardinality at most $(\frac{C_4}{\delta})^{\delta k_j} (\frac{N}{k_j})^{2\delta k_j} \leq (\frac{C_4 N}{\delta k_j})^{2\delta k_j}$ (such a net exists in view of Lemma 4). Similarly, we let $\mathcal{N}'_j \subset \mathbb{R}^{I^c}$ be a support-preserving $\frac{k_j}{N}$ -net in the set of δk_j -sparse unit vectors in \mathbb{R}^{I^c} , with $|\mathcal{N}'_j| \leq (\frac{C_4 N}{\delta k_j})^{2\delta k_j}$. Now, in view of Lemma 12, we have

$$\begin{aligned} g(k_j, \mathcal{C}, I) &\leq g(k_{j+1}, \mathcal{C}, I) + 2C_{11}\delta^{-2} \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \\ &\quad + C_{11}\sqrt{k_j}\delta^{-2} \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq \delta k_j, \\ \text{supp } y \subset I \cap \mathcal{C}}} (\lfloor k_{j+1}/4 \rfloor)\text{-} \max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}| \\ &\quad + C_{11}\sqrt{k_j}\delta^{-2} \sup_{\substack{z \in \mathbb{S}^{N-1}, \\ |\text{supp } z| \leq \delta k_j, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} (\lfloor k_{j+1}/4 \rfloor)\text{-} \max_{\ell \in I \cap \mathcal{C}} |W_{z,\ell}|. \end{aligned}$$

Applying Lemma 13 with $r := \lfloor k_{j+1}/4 \rfloor$, $\rho := \frac{k_j}{N}$, $h := \lfloor \delta k_j \rfloor$ and a $|I^c \cap \mathcal{C}| \times |I \cap \mathcal{C}|$ matrix $T = (\langle X_i, X_j \rangle) ((i, j) \in (I^c \cap \mathcal{C}) \times (I \cap \mathcal{C}))$, and using the definition of W 's (3), we get

$$\begin{aligned} &\sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq \delta k_j, \\ \text{supp } y \subset I \cap \mathcal{C}}} (\lfloor k_{j+1}/4 \rfloor)\text{-} \max_{\ell \in I^c \cap \mathcal{C}} |W_{y,\ell}| \\ &\leq 2 \sup_{u \in \mathcal{N}_j} (\lfloor \lfloor k_{j+1}/4 \rfloor / 2 \rfloor)\text{-} \max_{\ell \in I^c} |W_{u,\ell}| + \frac{4\frac{k_j}{N}}{\sqrt{\lfloor k_{j+1}/4 \rfloor}} \sup_{\substack{y \in \mathbb{S}^{N-1}, \\ |\text{supp } y| \leq \delta k_j, \\ \text{supp } y \subset I \cap \mathcal{C}}} \left(\sum_{i=1}^{\lfloor k_{j+1}/4 \rfloor} (i)\text{-} \max_{\ell \in I^c \cap \mathcal{C}} W_{y,\ell}^2 \right)^{1/2} \\ &\leq 2 \sup_{u \in \mathcal{N}_j} (\lfloor \lfloor k_{j+1}/4 \rfloor / 2 \rfloor)\text{-} \max_{\ell \in I^c} |W_{u,\ell}| + \frac{4\frac{k_j}{N}}{\sqrt{\lfloor k_{j+1}/4 \rfloor}} g(k_{j+1}, \mathcal{C}, I). \end{aligned}$$

Carrying out analogous estimate for

$$\sup_{\substack{z \in \mathbb{S}^{N-1}, \\ |\text{supp } z| \leq \delta k_j, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} (\lfloor k_{j+1}/4 \rfloor)\text{-} \max_{\ell \in I \cap \mathcal{C}} |W_{z,\ell}|,$$

we obtain

$$\begin{aligned} g(k_j, \mathcal{C}, I) &\leq \left(1 + \frac{32C_{11}k_j}{\delta^2 N}\right) g(k_{j+1}, \mathcal{C}, I) + 2C_{11}\delta^{-2} \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \\ &\quad + 2C_{11}\sqrt{k_j}\delta^{-2} \sup_{u \in \mathcal{N}_j} (\lfloor k_{j+1}/16 \rfloor)\text{-} \max_{\ell \in I^c} |W_{u,\ell}| \\ &\quad + 2C_{11}\sqrt{k_j}\delta^{-2} \sup_{v \in \mathcal{N}'_j} (\lfloor k_{j+1}/16 \rfloor)\text{-} \max_{\ell \in I} |W_{v,\ell}|. \end{aligned}$$

Note that, by the restrictions on N ,

$$\prod_{j=0}^t \left(1 + \frac{32C_{11}k_j}{\delta^2 N}\right) \leq \exp\left(\sum_{j=0}^t \frac{32C_{11}k_j}{\delta^2 N}\right) \leq \exp\left(\frac{128C_{11}k}{\delta^2 N}\right) \leq \exp(1).$$

Hence, recursively applying the above estimate for $g(k_j, \mathcal{C}, I)$ for all $0 \leq j < t$, we obtain

$$\begin{aligned} \exp(-1)g(k, \mathcal{C}, I) &\leq g(k_t, \mathcal{C}, I) + 2C_{11}\delta^{-2} \log_2(k) \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \\ &\quad + 2C_{11}\delta^{-2} \sum_{j=0}^{t-1} \sqrt{k_j} \sup_{u \in \mathcal{N}_j} (\lfloor k_{j+1}/16 \rfloor) \max_{\ell \in I^c} |W_{u,\ell}| \\ &\quad + 2C_{11}\delta^{-2} \sum_{j=0}^{t-1} \sqrt{k_j} \sup_{v \in \mathcal{N}'_j} (\lfloor k_{j+1}/16 \rfloor) \max_{\ell \in I} |W_{v,\ell}|. \end{aligned}$$

It remains to note that the quantity $g(k_t, \mathcal{C}, I)$ can be estimated as

$$\begin{aligned} g(k_t, \mathcal{C}, I) &\leq \sup_{\substack{y \in S^{N-1}, \\ |\text{supp } y| \leq k_t, \\ \text{supp } y \subset I \cap \mathcal{C}}} \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq k_t, \\ \text{supp } z \subset I^c \cap \mathcal{C}}} \sum_{i=1}^N \sum_{j=1}^N |y_i z_j \langle X_i, X_j \rangle| \\ &\leq \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \sup_{\substack{y \in S^{N-1}, \\ |\text{supp } y| \leq k_t}} \sup_{\substack{z \in S^{N-1}, \\ |\text{supp } z| \leq k_t}} \sum_{i,j=1}^N |y_i z_j| \\ &= k_t \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| \\ &\leq 48\delta^{-2} \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle|. \end{aligned}$$

□

5 Quadratic forms — Probabilistic estimates

In this section, we apply the deterministic bounds from Section 4 to obtain estimates for the tail distribution of quantity $f(n, [N])$ defined in (1). We always assume that X is an n -dimensional centered isotropic vector; X_1, X_2, \dots, X_N are its independent copies, and additionally suppose that $\sup_{y \in S^{n-1}} \mathbb{E} |\langle X, y \rangle|^p \leq B$ for some $p > 2$ and $B \geq 1$.

Let us start with the following corollary of Proposition 14:

Proposition 15. *There is a sufficiently large universal constant C_{15} with the following property: Let $I \subset [N]$ be fixed and $\mathcal{C} \subset [N]$ be random, and let $n, N > 1$ with $\log \frac{N}{n} \geq C_{15} \max(1, 1/(p-2))$. Then we have*

$$g(n, \mathcal{C}, I) \leq C_{15} \log^2 \frac{N}{n} \log(n) \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| + \frac{C_{15} p B^{1/p}}{p-2} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n} \right)^{1/p} \sqrt{f(n, [N])}$$

with probability at least $1 - \frac{1}{N^3}$.

Proof. First, consider the case when $n < 24 \log^2 \frac{N}{n}$. Then a crude deterministic bound on $g(n, \mathcal{C}, I)$ gives

$$g(n, \mathcal{C}, I) \leq n \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| < 24 \log^2 \frac{N}{n} \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle|,$$

and we get the statement.

For the rest of the proof, we assume that $n \geq 24 \log^2 \frac{N}{n}$. Define $\delta := \frac{1}{\log(N/n)}$, $t := \lfloor \log_2 \frac{\delta^2 n}{24} \rfloor$ and $k_j := \lfloor n/2^j \rfloor$ ($j = 0, 1, \dots, t$). We can assume that $\log(N/n)$ is sufficiently large, so that the conditions of Proposition 14 are satisfied. Fix for a moment any $0 \leq j < t$ and consider the quantity

$$\sup_{u \in \mathcal{N}_j} (\lfloor k_{j+1}/16 \rfloor)\text{-}\max_{\ell \in I^c} |W_{u,\ell}|,$$

where $\mathcal{N}_j \subset \mathbb{R}^I$ is defined in Proposition 14. Fix any $u \in \mathcal{N}_j$. Note that, conditioned on a realization of vectors X_i ($i \in I$), the quantities $W_{u,\ell} = \langle X_\ell, \sum_{i \in I} u_i X_i \rangle$ ($\ell \in I^c$) are jointly independent. Moreover, in view of the moment assumptions on X , the conditional expectation of $|W_{u,\ell}|^p$ given X_i ($i \in I$), satisfies

$$\mathbb{E}(|W_{u,\ell}|^p \mid X_i, i \in I) \leq B \left\| \sum_{i \in I} u_i X_i \right\|^p \leq B(f(n, [N]))^{p/2}.$$

Applying Lemma 8 to $|W_{u,\ell}|$'s with $\tau_j := (32eB)^{1/p} \sqrt{f(n, [N])} \left(\frac{N}{k_{j+1}}\right)^{p^{-1}(1+256\delta)}$, we get

$$\begin{aligned} \mathbb{P}\left\{ (\lfloor k_{j+1}/16 \rfloor)\text{-}\max_{\ell \in I^c} |W_{u,\ell}| \geq \tau_j \right\} &\leq \left(\frac{eBf(n, [N])^{p/2} N}{\tau_j^p \lfloor k_{j+1}/16 \rfloor} \right)^{\lfloor k_{j+1}/16 \rfloor} \\ &\leq \left(\frac{k_{j+1}}{N} \right)^{256\delta \lfloor k_{j+1}/16 \rfloor} \\ &\leq \left(\frac{k_{j+1}}{N} \right)^{4\delta k_j}. \end{aligned}$$

Now, taking the union bound over all $u \in \mathcal{N}_j$, we get

$$\mathbb{P}\left\{ \sup_{u \in \mathcal{N}_j} (\lfloor k_{j+1}/16 \rfloor)\text{-}\max_{\ell \in I^c} |W_{u,\ell}| \geq \tau_j \right\} \leq \left(\frac{k_{j+1}}{N} \right)^{4\delta k_j} |\mathcal{N}_j| \leq \left(\frac{C_4 k_j}{\delta N} \right)^{2\delta k_j}.$$

We can assume that $N \geq C_4^2 \delta^{-2} k_j$, so that

$$\left(\frac{C_4 k_j}{\delta N} \right)^{2\delta k_j} \leq \left(\frac{k_j}{N} \right)^{\delta k_j} \leq \left(\frac{k_t}{N} \right)^{\delta k_t} \ll \frac{1}{N^4}.$$

Thus,

$$\mathbb{P}\left\{ \sup_{u \in \mathcal{N}_j} (\lfloor k_{j+1}/16 \rfloor)\text{-}\max_{\ell \in I^c} |W_{u,\ell}| \geq \tau_j \right\} \leq \frac{1}{N^4}.$$

Summing up over j , repeating the same argument for nets \mathcal{N}'_j and applying Proposition 14, we get

$$g(n, \mathcal{C}, I) \leq C_{14} \delta^{-2} \log(n) \max_{i \neq j \in \mathcal{C}} |\langle X_i, X_j \rangle| + 2C_{14} \delta^{-2} \sum_{j=0}^{t-1} \sqrt{k_j} \tau_j$$

with probability at least $1 - \frac{1}{N^3}$. It remains to note that for some constant $C > 0$ the sum $\sum_{j=0}^{t-1} \sqrt{k_j} \tau_j$ can be estimated as

$$\sum_{j=0}^{t-1} \sqrt{k_j} \tau_j \leq C B^{1/p} \sqrt{n} \left(\frac{N}{n} \right)^{1/p} \sqrt{f(n, [N])} \sum_{j=0}^{t-1} 2^{j/p + 256\delta j/p - j/2},$$

and for a large enough constant C_{15} , the condition $\delta^{-1} = \log \frac{N}{n} \geq C_{15}/(p-2)$ implies that $\sum_{j=0}^{t-1} 2^{j/p + 256\delta j/p - j/2} \leq \sum_{j=0}^{t-1} 2^{j/(2p) - j/4} \leq \frac{\tilde{C}_p}{p-2}$. \square

Lemma 16. *Assume that $n, N > 1$ satisfy $\log \frac{N}{n} \geq C_{15} \max(1, 1/(p-2))$. Let $H > 0$, $m \in \mathbb{N}$, and let \mathcal{C}_m^H be the class from the coloring of X_1, X_2, \dots, X_N with threshold H . Then for a universal constant C_{16} we have*

$$\begin{aligned} f(n, \mathcal{C}_m^H) &\leq \max_{i \leq N} \|X_i\|^2 + C_{16} H \log^2 \frac{N}{n} \log(n) \max_{i \leq N} \|X_i\| \\ &\quad + \frac{C_{16} p B^{1/p}}{p-2} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n} \right)^{1/p} \sqrt{f(n, [N])} \end{aligned}$$

with probability at least $1 - \frac{1}{N^2}$.

Proof. Recall that

$$f(n, \mathcal{C}_m^H) \leq \max_{i \leq N} \|X_i\|^2 + 2^{-N+2} \sum_{I \subset [N]} g(n, \mathcal{C}_m^H, I). \quad (6)$$

Note that by Proposition 15, together with the definition of the class \mathcal{C}_m^H , we have for any $I \subset [N]$:

$$\begin{aligned} g(n, \mathcal{C}_m^H, I) &> C_{15} \log^2 \frac{N}{n} \log(n) \max_{i \neq j \in \mathcal{C}_m^H} |\langle X_i, X_j \rangle| \\ &\quad + \frac{C_{15} p B^{1/p}}{p-2} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n} \right)^{1/p} \sqrt{f(n, [N])} \end{aligned} \quad (7)$$

with probability at most $\frac{1}{N^3}$, whence

$$\mathbb{E} |\{I \subset [N] : g(n, \mathcal{C}_m^H, I) \text{ satisfies (7)}\}| \leq \frac{2^N}{N^3}.$$

Thus, by Markov's inequality,

$$\mathbb{P}\{g(n, \mathcal{C}_m^H, I) \text{ satisfies (7) for at least } 2^N/N \text{ subsets } I\} \leq \frac{1}{N^2}.$$

At the same time, a crude deterministic bound for $g(n, \mathcal{C}_m^H, I)$ gives

$$g(n, \mathcal{C}_m^H, I) \leq N \max_{i \neq j \in \mathcal{C}_m^H} |\langle X_i, X_j \rangle| \leq N H \max_{i \leq N} \|X_i\| \quad \text{for all } I \subset [N].$$

Combining the estimates, we obtain

$$\begin{aligned} \mathbb{P}\left\{\sum_{I \subset [N]} g(n, \mathcal{C}_m^H, I) \leq 2C_{15}2^N H \log^2 \frac{N}{n} \log(n) \max_{i \leq N} \|X_i\| \right. \\ \left. + \frac{C_{15}2^N p B^{1/p}}{p-2} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n}\right)^{1/p} \sqrt{f(n, [N])}\right\} \geq 1 - \frac{1}{N^2}. \end{aligned}$$

Thus, applying (6), we get

$$\begin{aligned} f(n, \mathcal{C}_m^H) \leq \max_{i \leq N} \|X_i\|^2 + 8C_{15} H \log^2 \frac{N}{n} \log(n) \max_{i \leq N} \|X_i\| \\ + \frac{4C_{15} p B^{1/p}}{p-2} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n}\right)^{1/p} \sqrt{f(n, [N])} \end{aligned}$$

with probability at least $1 - \frac{1}{N^2}$. \square

Combining the last statement with the proposition from Section 3, we obtain the main result of this section.

Proposition 17. *There is a non-increasing function $\nu : (2, \infty) \rightarrow \mathbb{R}_+$ with the following property: Let $N, n > 1$, $p > 2$, $B \geq 1$, and assume that $\log \frac{N}{n} \geq C_{15} \max(1, 1/(p-2))$. Let, as before, X be a centered n -dimensional isotropic random vector, X_1, X_2, \dots, X_N be its independent copies, and assume that $\sup_{a \in S^{n-1}} \mathbb{E}|\langle X, a \rangle|^p \leq B$. Finally, let $f(n, [N])$ be defined by (1). Then*

$$f(n, [N]) \leq \nu(p) \max_{i \leq N} \|X_i\|^2 + \nu(p) B^{2/p} n \left(\frac{N}{n}\right)^{2/p} \log^4 \frac{N}{n}$$

with probability at least $1 - \frac{2}{n^2}$.

Proof. If $n < \left(\frac{8p}{p-2}\right)^{8p/(p-2)}$ then a crude deterministic bound for $f(n, [N])$ gives

$$f(n, [N]) \leq n \max_{i \leq N} \|X_i\|^2 \leq \left(\frac{8p}{p-2}\right)^{8p/(p-2)} \max_{i \leq N} \|X_i\|^2,$$

and we obtain the statement.

Otherwise, we have

$$n \geq \left(\frac{8p}{p-2}\right)^{8p/(p-2)}, \quad (8)$$

and both n and N satisfy the assumptions of Proposition 15. Define

$$H := (BN)^{1/p} n^{1/2-1/p} / \log(n)$$

and let $\chi := \lceil \frac{8+2p}{p-2} \rceil$. Then, by Lemma 16 and the definition of H , we have

$$\begin{aligned} \chi^{-1} \sum_{m=1}^{\chi} f(n, \mathcal{C}_m^H) \leq \max_{i \leq N} \|X_i\|^2 + C_{16} B^{1/p} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n}\right)^{1/p} \max_{i \leq N} \|X_i\| \\ + \frac{C_{16} p B^{1/p}}{p-2} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n}\right)^{1/p} \sqrt{f(n, [N])} \end{aligned} \quad (9)$$

with probability at least $1 - \frac{\chi}{N^2}$. Now, recall that $\mathcal{C}_m^H = \emptyset$ for all $m > \chi(\mathcal{G}_H)$, where \mathcal{G}_H is the graph defined in Section 3, and $\chi(\mathcal{G}_H)$ is its chromatic number. By Proposition 10, we have $\chi(\mathcal{G}_H) \leq \chi$ with probability at least $1 - (BNH^{-p})^\chi n^{p/2}$. Note that, by the assumption (8) on n , we have

$$n^{1/2-1/p}/\log(n) \geq n^{1/4-1/(2p)},$$

whence

$$(BNH^{-p})^\chi = (n^{p/2-1}/\log^p(n))^{-\chi} \leq n^{\chi/2-p\chi/4} \leq \frac{1}{n^{2+p/2}}.$$

Thus, $\chi(\mathcal{G}_H) \leq \chi$ with probability at least $1 - \frac{1}{n^2}$. Note that by the definition of $f(n, \mathcal{C})$ and the partition $\{\mathcal{C}_m^H\}_{m=1}^N$, and by the Cauchy–Schwarz inequality, we have

$$f(n, [N]) \leq \sum_{m=1}^{\chi(\mathcal{G}_H)} f(n, \mathcal{C}_m^H)$$

deterministically. Therefore, in view of the above estimate of the chromatic number, we have

$$f(n, [N]) \leq \sum_{m=1}^{\chi} f(n, \mathcal{C}_m^H)$$

with probability at least $1 - \frac{1}{n^2}$. Together with the probability bound for (9), it yields

$$\begin{aligned} \chi^{-1} f(n, [N]) &\leq \max_{i \leq N} \|X_i\|^2 + C_{16} B^{1/p} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n}\right)^{1/p} \max_{i \leq N} \|X_i\| \\ &\quad + \frac{C_{16} p B^{1/p}}{p-2} \log^2 \frac{N}{n} \sqrt{n} \left(\frac{N}{n}\right)^{1/p} \sqrt{f(n, [N])} \end{aligned}$$

with probability at least $1 - \frac{\chi}{N^2} - \frac{1}{n^2}$. Solving the inequality, we obtain the result. \square

6 Proof of Theorem 1

The contents of this section is to a large extent based on arguments from papers [1, 20, 10]. Let us emphasize that the new ingredients — the Sparsifying Lemma 11 and the coloring of the sample from Section 3 — were employed to bound the quantity $f(n, [N])$, whereas transition from those bounds to estimating the extreme singular values of the sample covariance matrix is well understood and covered in literature. Nevertheless, we prefer to include all the proofs for completeness.

We start with estimating the Euclidean norm of a tail of a random vector with independent coordinates.

Lemma 18. *Let $n, N \in \mathbb{N}$ with $n \leq N$, and let $Y = (Y_1, Y_2, \dots, Y_N)$ be a vector of independent non-negative random variables such that $\mathbb{E}Y_i^q \leq B$ ($i = 1, 2, \dots, N$) for some $q > 1$ and $B \geq 1$. Then*

$$\left(\sum_{i=n+1}^N (i) \cdot \max_{\ell \in [N]} Y_\ell^2 \right)^{1/2} \leq C_{18} B^{1/q} \sqrt{n} \left(\frac{N}{n}\right)^{1/\min(q, 2)}$$

with probability at least $1 - \exp(-3n)$. Here, $C_{18} > 0$ is a sufficiently large universal constant.

Proof. Set $M := \left(\frac{e^5 BN}{n}\right)^{1/q}$. By Markov's inequality,

$$\begin{aligned} \mathbb{P}\{|\{i \leq N : Y_i \geq M\}| > n\} &= \mathbb{P}\{|\{i \leq N : Y_i^q \geq e^5 BN/n\}| > n\} \\ &\leq \binom{N}{n} \left(\frac{n}{e^5 N}\right)^n \\ &\leq \exp(-4n). \end{aligned}$$

We define $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_N)$ as a vector of truncations of Y_i 's: for every point ω of the probability space, we set

$$\tilde{Y}_i(\omega) := \begin{cases} Y_i(\omega), & \text{if } Y_i(\omega) \leq M; \\ M, & \text{otherwise.} \end{cases}$$

Then, from the above estimate,

$$\begin{aligned} \mathbb{P}\left\{\left(\sum_{i=n+1}^N (i)\text{-}\max_{\ell \in [N]} \tilde{Y}_\ell^2\right)^{1/2} < \left(\sum_{i=n+1}^N (i)\text{-}\max_{\ell \in [N]} Y_\ell^2\right)^{1/2}\right\} &= \mathbb{P}\{|\{i \leq N : Y_i > M\}| > n\} \\ &\leq \exp(-4n). \end{aligned}$$

Now, we estimate the Euclidean norm of \tilde{Y} using the Laplace transform. We set $\lambda := \frac{1}{M^2} = \left(\frac{n}{e^5 BN}\right)^{2/q}$. We have

$$\begin{aligned} \mathbb{E} \exp(\lambda \|\tilde{Y}\|^2) &= \prod_{i=1}^N \mathbb{E} \exp(\lambda \tilde{Y}_i^2) \\ &= \prod_{i=1}^N \left(1 + \int_1^{\exp(\lambda M^2)} \mathbb{P}\{\exp(\lambda \tilde{Y}_i^2) \geq \tau\} d\tau\right) \\ &\leq \prod_{i=1}^N \left(1 + \int_1^e \mathbb{P}\left\{\tilde{Y}_i^2 \geq \frac{\tau - 1}{e\lambda}\right\} d\tau\right) \\ &\leq \prod_{i=1}^N \left(1 + e\lambda \int_0^{(e-1)/(e\lambda)} \mathbb{P}\{\tilde{Y}_i^2 \geq u\} du\right) \\ &\leq \prod_{i=1}^N (1 + e\lambda \mathbb{E} \tilde{Y}_i^2). \end{aligned}$$

First, assume that $q \geq 2$. Then $\mathbb{E} \tilde{Y}_i^2 \leq B^{2/q}$, and we get

$$\mathbb{E} \exp(\lambda \|\tilde{Y}\|^2) \leq (1 + eB^{2/q}\lambda)^N \leq \exp(eB^{2/q}\lambda N).$$

Otherwise, if $q < 2$ then $\mathbb{E} \tilde{Y}_i^2 \leq M^{2-q} \mathbb{E} \tilde{Y}_i^q \leq B\lambda^{q/2-1}$, whence

$$\mathbb{E} \exp(\lambda \|\tilde{Y}\|^2) \leq (1 + eB\lambda^{q/2})^N \leq \exp(eB\lambda^{q/2}N).$$

Thus, denoting $r := \min(q, 2)$, we get

$$\mathbb{E} \exp(\lambda \|\tilde{Y}\|^2) \leq \exp(e B^{2/q-2/r+1} \lambda^{r/2} N).$$

Hence, by Markov's inequality,

$$\begin{aligned} \mathbb{P}\{\|\tilde{Y}\| \geq e^6 B^{1/q} N^{1/r} n^{1/2-1/r}\} &\leq \exp(e B^{2/q-2/r+1} \lambda^{r/2} N - e^{12} B^{2/q} \lambda N^{2/r} n^{1-2/r}) \\ &\leq \exp(-(e^7 - e) B^{r/q} \lambda^{r/2} N) \\ &\leq \exp(-4n). \end{aligned}$$

Finally, we get

$$\begin{aligned} &\mathbb{P}\left\{\left(\sum_{i=n+1}^N (i)\text{-}\max_{\ell \in [N]} Y_\ell^2\right)^{1/2} > e^6 B^{1/q} N^{1/r} n^{1/2-1/r}\right\} \\ &\leq \mathbb{P}\left\{\left(\sum_{i=n+1}^N (i)\text{-}\max_{\ell \in [N]} \tilde{Y}_\ell^2\right)^{1/2} < \left(\sum_{i=n+1}^N (i)\text{-}\max_{\ell \in [N]} Y_\ell^2\right)^{1/2}\right\} + \mathbb{P}\{\|\tilde{Y}\| > e^6 B^{1/q} N^{1/r} n^{1/2-1/r}\} \\ &\leq 2 \exp(-4n) \\ &\leq \exp(-3n). \end{aligned}$$

□

Remark 1. The above lemma is similar, but not identical to [10, Lemma 4.4], which was proved under slightly different assumptions, and using different arguments.

Proposition 19. *Let X_1, X_2, \dots, X_N be i.i.d. centered n -dimensional isotropic random vectors, and assume that for some $p > 2$ and $B \geq 1$ we have*

$$\mathbb{E} |\langle X_i, a \rangle|^p \leq B$$

for all $a \in S^{n-1}$. Further, let r_1, r_2, \dots, r_N be Rademacher (± 1) random variables jointly independent with X_1, X_2, \dots, X_N . Then

$$\sup_{a \in S^{n-1}} \left| \sum_{i=1}^N r_i \langle X_i, a \rangle^2 \right| \leq C_{19} B^{2/p} n \left(\frac{N}{n} \right)^{2/\min(p,4)} + 2f(n, [N])$$

with probability at least $1 - 2^{-n}$. Here, f is defined by (1), and $C_{19} > 0$ is a universal constant.

Proof. Define a random operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$Ta := \sum_{i=1}^N r_i \langle X_i, a \rangle X_i, \quad a \in S^{n-1}.$$

Let \mathcal{N} be a Euclidean $1/4$ -net on S^{n-1} of cardinality at most 9^n . Then, applying Lemma 5, we obtain

$$\sup_{a \in S^{n-1}} |\langle Ta, a \rangle| \leq 2 \sup_{a \in \mathcal{N}} |\langle Ta, a \rangle| = 2 \sup_{a \in \mathcal{N}} \left| \sum_{i=1}^N r_i \langle X_i, a \rangle^2 \right|.$$

Next, for every $a \in S^{n-1}$ let σ_a be a random permutation on $[N]$ measurable with respect to the σ -algebra generated by X_1, X_2, \dots, X_N , such that

$$\langle X_{\sigma_a(1)}, a \rangle^2 \geq \langle X_{\sigma_a(2)}, a \rangle^2 \geq \dots \geq \langle X_{\sigma_a(N)}, a \rangle^2.$$

Thus, $(i)\text{-}\max_{\ell \in [N]} \langle X_\ell, a \rangle^2 = \langle X_{\sigma_a(i)}, a \rangle^2$ for all $i = 1, 2, \dots, N$. We have

$$\begin{aligned} \sup_{a \in \mathcal{N}} \left| \sum_{i=1}^N r_i \langle X_i, a \rangle^2 \right| &\leq \sup_{a \in S^{n-1}} \sum_{i=1}^n (i)\text{-}\max_{\ell \in [N]} \langle X_\ell, a \rangle^2 + \sup_{a \in \mathcal{N}} \left| \sum_{i=n+1}^N r_{\sigma_a(i)} \langle X_{\sigma_a(i)}, a \rangle^2 \right| \\ &= f(n, [N]) + \sup_{a \in \mathcal{N}} \left| \sum_{i=n+1}^N r_{\sigma_a(i)} \langle X_{\sigma_a(i)}, a \rangle^2 \right|. \end{aligned}$$

In view of the fact that σ_a is independent from r_1, r_2, \dots, r_N , it remains to prove that

$$\sup_{a \in \mathcal{N}} \left| \sum_{i=n+1}^N r_i (i)\text{-}\max_{\ell \in [N]} \langle X_\ell, a \rangle^2 \right| \leq C B^{2/p} n \left(\frac{N}{n} \right)^{2/\min(p,4)}$$

with probability at least $1 - 2^{-n}$ for a sufficiently large universal constant $C > 0$. Fix for a moment $a \in \mathcal{N}$ and define a random vector $Z^a \in \mathbb{R}^{[N] \setminus [n]}$ by

$$Z_i^a := (i)\text{-}\max_{\ell \in [N]} \langle X_\ell, a \rangle^2, \quad i = n+1, \dots, N.$$

Note that Z^a and r_1, r_2, \dots, r_N are jointly independent. Applying Hoeffding's inequality, we get

$$\left| \sum_{i=n+1}^N r_i Z_i^a \right| \leq 4\sqrt{n} \|Z^a\|$$

with probability at least $1 - 2 \exp(-8n)$ (see Lemma 6). At the same time, applying Lemma 18 with $Y_i := \langle X_i, a \rangle^2$ ($i = 1, 2, \dots, N$) and $q := p/2$, we get

$$\mathbb{P} \left\{ \|Z^a\| > C_{18} B^{2/p} \sqrt{n} \left(\frac{N}{n} \right)^{2/\min(p,4)} \right\} \leq \exp(-3n).$$

Combining the last two estimates, we obtain

$$\begin{aligned} \mathbb{P} \left\{ \left| \sum_{i=n+1}^N r_i (i)\text{-}\max_{\ell \in [N]} \langle X_\ell, a \rangle^2 \right| > 4C_{18} B^{2/p} n \left(\frac{N}{n} \right)^{2/\min(p,4)} \right\} &\leq 2 \exp(-8n) + \exp(-3n) \\ &\leq 18^{-n}. \end{aligned}$$

Taking the union bound over all $a \in \mathcal{N}$, we obtain the desired result. \square

In the next statement, we combine Proposition 19 with a standard symmetrization argument. Let us note once more that at this point our proof essentially follows the argument of [20] and [10].

Proposition 20. Let X_1, X_2, \dots, X_N be i.i.d. centered isotropic random vectors, such that for some $p > 2$ and $B \geq 1$ we have

$$\sup_{a \in \mathbb{S}^{n-1}} \mathbb{E} |\langle X_i, a \rangle|^p \leq B.$$

Then

$$\sup_{a \in \mathbb{S}^{n-1}} \left| \sum_{i=1}^N \langle X_i, a \rangle^2 - N \right| \leq C_{20} B^{2/p} n \left(\frac{N}{n} \right)^{2/\min(p,4)} + 4f(n, [N])$$

with probability at least $1 - 2^{2-n}$. Here, $C_{20} > 0$ is a universal constant.

Proof. Let X'_1, X'_2, \dots, X'_N be jointly independent copies of X_1, X_2, \dots, X_N , and for every $a \in \mathbb{S}^{n-1}$ denote

$$\xi_a := \sum_{i=1}^N \langle X_i, a \rangle^2 - N; \quad \xi'_a := \sum_{i=1}^N \langle X'_i, a \rangle^2 - N.$$

Then the variable

$$\sup_{a \in \mathbb{S}^{n-1}} |\xi_a - \xi'_a| = \sup_{a \in \mathbb{S}^{n-1}} \left| \sum_{i=1}^N (\langle X_i, a \rangle^2 - \langle X'_i, a \rangle^2) \right|$$

has the same distribution as

$$\sup_{a \in \mathbb{S}^{n-1}} \left| \sum_{i=1}^N r_i (\langle X_i, a \rangle^2 - \langle X'_i, a \rangle^2) \right|,$$

where r_1, r_2, \dots, r_N are Rademacher random variables jointly independent with the vectors X_i, X'_i . Hence, for any $t \geq 0$ we have

$$\mathbb{P} \left\{ \sup_{a \in \mathbb{S}^{n-1}} |\xi_a - \xi'_a| \geq t \right\} \leq 2\mathbb{P} \left\{ \sup_{a \in \mathbb{S}^{n-1}} \left| \sum_{i=1}^N r_i \langle X_i, a \rangle^2 \right| \geq \frac{t}{2} \right\}.$$

Applying Proposition 19, we obtain

$$\mathbb{P} \left\{ \sup_{a \in \mathbb{S}^{n-1}} |\xi_a - \xi'_a| > 2C_{19} B^{2/p} n \left(\frac{N}{n} \right)^{2/\min(p,4)} + 4f(n, [N]) \right\} \leq 2^{1-n}. \quad (10)$$

Finally, note that for any $a \in \mathbb{S}^{n-1}$ we have

$$\begin{aligned} \mathbb{E} |\xi_a| &\leq \mathbb{E} |\xi_a - \xi'_a| \\ &= \mathbb{E} \left| \sum_{i=1}^N r_i (\langle X_i, a \rangle^2 - \langle X'_i, a \rangle^2) \right| \\ &\leq 2\mathbb{E} \left| \sum_{i=1}^N r_i \langle X_i, a \rangle^2 \right| \\ &\leq 2\mathbb{E} \left(\sum_{i=1}^N \langle X_i, a \rangle^4 \right)^{1/2}, \end{aligned}$$

whence, by the Minkowski inequality,

$$\begin{aligned}\mathbb{E}|\xi_a| &\leq 2\mathbb{E}\left(\sum_{i=1}^N |\langle X_i, a \rangle|^{\min(p,4)}\right)^{2/\min(p,4)} \\ &\leq 2\left(\sum_{i=1}^N \mathbb{E}|\langle X_i, a \rangle|^{\min(p,4)}\right)^{2/\min(p,4)} \\ &\leq 2B^{2/p}N^{2/\min(p,4)}.\end{aligned}$$

Therefore, by Markov's inequality

$$\text{Med}|\xi_a| \leq 4B^{2/p}N^{2/\min(p,4)}, \quad a \in S^{n-1}.$$

Combining this with a standard estimate

$$\mathbb{P}\left\{\sup_{a \in S^{n-1}} |\xi_a| \geq t + M\right\} \leq 2\mathbb{P}\left\{\sup_{a \in S^{n-1}} |\xi_a - \xi'_a| \geq t\right\}, \quad t > 0,$$

where $M := \sup_{a \in S^{n-1}} \text{Med}|\xi'_a|$, and with (10), we obtain the result. \square

Proof of Theorem 1. Let X be a centered isotropic random vector in \mathbb{R}^n , such that

$$\sup_{a \in S^{n-1}} \mathbb{E}|\langle X, a \rangle|^p \leq B$$

for some $p > 2$ and $B \geq 1$. Also, let X_1, X_2, \dots, X_N be its independent copies. As before, we denote by Σ_N the sample covariance matrix for X_1, X_2, \dots, X_N , and by A_N — the $N \times n$ random matrix with rows X_1, X_2, \dots, X_N . If N is bounded by a function of p , we can apply a trivial estimate to get the result. So, further we assume that

$$N > 2 \exp(3C_{15} \max(1, 1/(p-2))).$$

First, suppose that $\log \frac{N}{n} \geq C_{15} \max(1, 1/(p-2))$. We have

$$\begin{aligned}\|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2} &= \sup_{a \in S^{n-1}} |\langle \Sigma_N a - a, a \rangle| \\ &= N^{-1} \sup_{a \in S^{n-1}} |\langle A_N^T A_N a, a \rangle - N| \\ &= N^{-1} \sup_{a \in S^{n-1}} \left| \sum_{i=1}^N \langle X_i, a \rangle^2 - N \right|.\end{aligned}$$

Hence, by Proposition 20, we have

$$\|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2} \leq C_{20} B^{2/p} n \left(\frac{N}{n}\right)^{2/\min(p,4)} + 4f(n, [N])$$

with probability at least $1 - 2^{2-n}$. Then apply Proposition 17.

Now, if $\log \frac{N}{n} < C_{15} \max(1, 1/(p-2))$ then it is enough to check that

$$\|(A_N)^T A_N\|_{2 \rightarrow 2} \leq \tilde{\nu}(p) \max_{i \leq N} \|X_i\|^2 + \tilde{\nu}(p) B^{2/p} N$$

with probability $\geq 1 - \frac{1}{n}$ for some non-increasing function $\nu(p) : (2, \infty) \rightarrow \mathbb{R}_+$. Set $n_0 := \lfloor N / \exp(C_{15} \max(1, 1/(p-2))) \rfloor$ and consider an arbitrary partition $\{J_m\}_{m=1}^{\lceil n/n_0 \rceil}$ of $[n]$ with $\max_m |J_m| \leq n_0$. For every $m \leq \lceil n/n_0 \rceil$, we let A_N^m be the $[N] \times J_m$ -submatrix of A_N . Note that the rows of A_N^m are isotropic (in \mathbb{R}^{J_m}), i.i.d., and satisfy the p -th moment condition for one-dimensional projections. Moreover, the ratio of the numbers of rows and columns in every matrix A_N^m satisfies the assumptions of the first part of the theorem. Hence, by the above argument, for any $m \leq \lceil n/n_0 \rceil$ we have

$$\|(A_N^m)^T A_N^m\|_{2 \rightarrow 2} \leq \nu'(p) \max_{i \leq N} \|X_i\|^2 + \nu'(p) B^{2/p} N$$

with probability at least $1 - \frac{2}{n_0^2} - 2^{2-n_0}$. Since

$$\|(A_N)^T A_N\|_{2 \rightarrow 2} \leq \sum_{m=1}^{\lceil n/n_0 \rceil} \|(A_N^m)^T A_N^m\|_{2 \rightarrow 2},$$

we obtain the result by taking the union bound.

Finally, we may use a standard linear algebraic argument to pass from isotropic distributions to all distributions from the class $\mathcal{F}(n, p, B)$. \square

Let us briefly discuss optimality of the result obtained. As we already mentioned, for $p = 4$ the log-factor which appears in our bound in Theorem 1, seems excessive. In the range $2 < p < 4$, the situation is more unclear to us. We do not know whether the estimate for the difference $\|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2}$ (for isotropic distributions) can be improved if we assume a strong concentration for the vector norm. Let us formulate the problem in a more precise form:

Problem. Let $2 < p < 4$ and assume that X is a centered n -dimensional isotropic random vector such that $\|X\| \leq C\sqrt{n}$ a.s. and $\sup_{y \in S^{n-1}} \mathbb{E}|\langle X, y \rangle|^p \leq C$ for a large universal constant $C > 0$. Let $N \geq n$ and let X_1, X_2, \dots, X_N be independent copies of X . As before, let Σ_N be the sample covariance matrix with respect to X_1, X_2, \dots, X_N . Is it true that

$$\|\Sigma_N - \text{Id}_n\|_{2 \rightarrow 2} \leq K \sqrt{\frac{n}{N}}$$

with probability close to one, where K depends only on p ?

Let us remark that an example of P. Yaskov [32] which provides *upper* bounds for the smallest eigenvalue of Σ_N for certain isotropic distributions, does not resolve the above problem in negative as an essential assumption in [32] is a growth condition on the vector norm.

Acknowledgement. I would like to thank Nicole Tomczak-Jaegermann for her support, Alexander Litvak for clarifying some arguments from [10], and Shahar Mendelson and Ramon van Handel for a fruitful discussion.

References

- [1] R. Adamczak, A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann, Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles, *J. Amer. Math. Soc.* **23** (2010), no. 2, 535–561. MR2601042
- [2] R. Adamczak, A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann, Sharp bounds on the rate of convergence of the empirical covariance matrix, *C. R. Math. Acad. Sci. Paris* **349** (2011), no. 3-4, 195–200. MR2769907
- [3] Z. D. Bai and Y. Q. Yin, Limit of the smallest eigenvalue of a large-dimensional sample covariance matrix, *Ann. Probab.* **21** (1993), no. 3, 1275–1294. MR1235416
- [4] P. J. Bickel and E. Levina, Covariance regularization by thresholding, *Ann. Statist.* **36** (2008), no. 6, 2577–2604. MR2485008
- [5] P. J. Bickel and E. Levina, Regularized estimation of large covariance matrices, *Ann. Statist.* **36** (2008), no. 1, 199–227. MR2387969
- [6] J. Bourgain, An improved estimate in the restricted isometry problem, in *Geometric aspects of functional analysis*, 65–70, Lecture Notes in Math., 2116, Springer, Cham. MR3364679
- [7] J. Bourgain, Random points in isotropic convex sets, in *Convex geometric analysis (Berkeley, CA, 1996)*, 53–58, Math. Sci. Res. Inst. Publ., 34, Cambridge Univ. Press, Cambridge. MR1665576
- [8] D. Chafaï, K. Tikhomirov, On the convergence of the extremal eigenvalues of empirical covariance matrices with dependence, Preprint. arXiv:1509.02231
- [9] O. N. Feldheim and S. Sodin, A universality result for the smallest eigenvalues of certain sample covariance matrices, *Geom. Funct. Anal.* **20** (2010), no. 1, 88–123. MR2647136
- [10] O. Guédon, A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann, On the interval of fluctuation of the singular values of random matrices, *J. Eur. Math. Soc. (JEMS)*, to appear.
- [11] O. Guédon, A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann, Restricted isometry property for random matrices with heavy-tailed columns, *C. R. Math. Acad. Sci. Paris* **352** (2014), no. 5, 431–434. MR3194251
- [12] I. Haviv, O. Regev, The Restricted Isometry Property of Subsampled Fourier Matrices, arXiv:1507.01768
- [13] W. Hoeffding, Probability inequalities for sums of bounded random variables, *J. Amer. Statist. Assoc.* **58** (1963), 13–30. MR0144363
- [14] I. M. Johnstone, On the distribution of the largest eigenvalue in principal components analysis, *Ann. Statist.* **29** (2001), no. 2, 295–327. MR1863961

- [15] R. Kannan, L. Lovász and M. Simonovits, Random walks and an $O^*(n^5)$ volume algorithm for convex bodies, *Random Structures Algorithms* **11** (1997), no. 1, 1–50. MR1608200
- [16] H. Kesten, A sharper form of the Doeblin-Lévy-Kolmogorov-Rogozin inequality for concentration functions, *Math. Scand.* **25** (1969), 133–144. MR0258095
- [17] V. Koltchinskii and S. Mendelson, Bounding the smallest singular value of a random matrix without concentration, *Int. Math. Res. Not. IMRN* **2015**, no. 23, 12991–13008. MR3431642
- [18] A. E. Litvak, A. Pajor, M. Rudelson, N. Tomczak-Jaegermann, Smallest singular value of random matrices and geometry of random polytopes, *Adv. Math.* **195** (2005), no. 2, 491–523. MR2146352
- [19] S. Mendelson and G. Paouris, On generic chaining and the smallest singular value of random matrices with heavy tails, *J. Funct. Anal.* **262** (2012), no. 9, 3775–3811. MR2899978
- [20] S. Mendelson and G. Paouris, On the singular values of random matrices, *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 4, 823–834. MR3191978
- [21] N. S. Pillai and J. Yin, Universality of covariance matrices, *Ann. Appl. Probab.* **24** (2014), no. 3, 935–1001. MR3199978
- [22] M. Rudelson, Random vectors in the isotropic position, *J. Funct. Anal.* **164** (1999), no. 1, 60–72. MR1694526
- [23] M. Rudelson and R. Vershynin, On sparse reconstruction from Fourier and Gaussian measurements, *Comm. Pure Appl. Math.* **61** (2008), no. 8, 1025–1045. MR2417886
- [24] M. Rudelson and R. Vershynin, Non-asymptotic theory of random matrices: extreme singular values, in *Proceedings of the International Congress of Mathematicians. Volume III*, 1576–1602, Hindustan Book Agency, New Delhi. MR2827856
- [25] M. Rudelson and R. Vershynin, Smallest singular value of a random rectangular matrix, *Comm. Pure Appl. Math.* **62** (2009), no. 12, 1707–1739. MR2569075
- [26] N. Srivastava and R. Vershynin, Covariance estimation for distributions with $2 + \varepsilon$ moments, *Ann. Probab.* **41** (2013), no. 5, 3081–3111. MR3127875
- [27] K. Tikhomirov, The limit of the smallest singular value of random matrices with i.i.d. entries, *Adv. Math.* **284** (2015), 1–20. MR3391069
- [28] K. E. Tikhomirov, The smallest singular value of random rectangular matrices with no moment assumptions on entries, *Israel J. Math.* **212** (2016), no. 1, 289–314. MR3504328
- [29] R. Vershynin, How close is the sample covariance matrix to the actual covariance matrix?, *J. Theoret. Probab.* **25** (2012), no. 3, 655–686. MR2956207

- [30] R. Vershynin, Introduction to the non-asymptotic analysis of random matrices, in *Compressed sensing*, 210–268, Cambridge Univ. Press, Cambridge. MR2963170
- [31] P. Yaskov, Lower bounds on the smallest eigenvalue of a sample covariance matrix, *Electron. Commun. Probab.* **19** (2014), no. 83, 10 pp. MR3291620
- [32] P. Yaskov, Sharp lower bounds on the least singular value of a random matrix without the fourth moment condition, *Electron. Commun. Probab.* **20** (2015), no. 44, 9 pp. MR3358966
- [33] Y. Q. Yin, Z. D. Bai and P. R. Krishnaiah, On the limit of the largest eigenvalue of the large-dimensional sample covariance matrix, *Probab. Theory Related Fields* **78** (1988), no. 4, 509–521. MR0950344